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On Axiomatic Characterization of Information-Theoretic Measures

Bhu Dev Sharma[1] and Inder Jeet Taneja[1]

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An axiomatic characterization of an information-theoretic quantity associated with a pair of probability distributions having the same number of elements has been given. This quantity, under additional suitable conditions, leads to Kullback's information and Kerridge's inaccuracy concepts. By modifying one of the axioms, the two-parameter generalization of these is obtained.

KEY WORDS: Information theory; measures of information; inaccuracy; statistical estimation; parametric generalization.

1. INTRODUCTION

Let D^n denote the set of P^n, where $P^n = (p_1,..., p_n), p_i \geqslant 0, \sum_{i=1}^n p_i \leqslant 1$, is an n-probability vector. Also let a subset $\bar{D}^n$ of D^n contain only those P^n for which $\sum_{i=1}^n p_i = 1$. Further, we shall denote by $\Delta_n = \bar{D}^n \times D^n$ the set of all ordered pairs $(P^n; Q^n)$, $P^n \in \bar{D}^n$ and $Q^n \in D^n$.

There are two information-theoretic measures associated with a pair of n-probability vectors which are of great significance in statistical estimation and physics. One is the measure of information known as Kullback's informa-

[1] Faculty of Mathematics, University of Delhi, Delhi, India.

tion or directed divergence[2,6–8,10] and is given by

$$ {}_1I_n(P^n; Q^n) = \sum_{i=1}^{n} p_i \log(p_i/q_i) \tag{1} $$

and the other is Kerridge's inaccurary[7–9] given by

$$ {}_2I_n(P^n; Q^n) = -\sum_{i=1}^{n} p_i \log q_i \tag{2} $$

Both these include Shannon's entropy as a particular case.

The object of this paper is to give an axiomatic characterization of a measure which jointly contains (1) and (2). Also, by taking further conditions, and these are essentially those which make them basically different from Shannon's entropy, we obtain these two measures separately. A suitable modification in one of the axioms which specifies the branching property of the measures has been used to study two-parameter generalizations of (1) and (2).

In what follows we shall assume that $0 \log 0 = 0 \log(0/0) = 0$ and all logarithms are considered to the base 2.

2. AXIOMS FOR INFORMATION MEASURES

We consider a mapping I_n of Δ_n into set of real numbers, i.e.,

$$ I_n: \quad \Delta_n \to R \text{ (reals)} $$

Our aim is to make the function I_n a measure of information for a pair of n-probability vectors. We assume the following axioms:

Axiom 1. (Symmetry): $I_4(P^4; Q^4)$ is symmetric for any permutation of elements in P^4 followed by the same permutation in Q^4.

Axiom 2. (Branching property):

$$ I_n(P^n; Q^n) - I_{n-1}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) $$
$$ = \phi(p_1, p_2; q_1, q_2), \qquad n = 3, 4, \ldots $$

Axiom 3. (Additivity):

$$ I_{2n}(P^n * R^2; Q^n * S^2) = I_n(P^n; Q^n) + I_2(R^2; S^2) $$

where $R^2 = (r, 1 - r)$ and $P^n * R^2 = (p_1 r, p_1(1 - r), \ldots, p_n r, p_n(1 - r))$, for $n = 2, 3$.

It is clear from Axiom 1 that the symmetry requirement is very limited as taken for $n = 4$; this is also the case with additivity in Axiom 3, where we use only I_2; the branching property is a natural adaptation of one taken by Faddeev (see Ref. 3) for characterizing Shannon's entropy.

Now we will give as lemmas some results based on the above axioms.

Lemma 1. The function I_n satisfying Axioms 1–3 is symmetric for every n.

Proof of this lemma follows exactly on the lines of Forte and Daroczy.[4]

Lemma 2. If I_n verifies Axioms 1–3 and we define

$$\psi_{r,s}(p; q) = \phi(pr, p(1 - r); qs, q(1 - s)) \tag{3}$$

then for each $r, s \in [0, 1]$, $\psi_{r,s}$ satisfies Cauchy's functional equation,

$$\psi_{r,s}(p_1 + p_2; q_1 + q_2) = \psi_{r,s}(p_1 : q_1) + \psi_{r,s}(p_2; q_2) \tag{4}$$

where $p_1, p_2, q_1, q_2 \geqslant 0; p_1 + p_2 \leqslant 1; q_1 + q_2 \leqslant 1.$

Proof. From Lemma 1 and Axiom 2, we get

$$I_{2n}(P^n * R^2; Q^n * S^2)$$
$$= I_n(P^n; Q^n) + \sum_{i=1}^{n} \phi(p_i r, p_i(1 - r); q_i s, q_i(1 - s)) \tag{5}$$

On the other hand, from Axiom 3 for $n = 2$ and 3, we have

$$I_{2n}(P^n * R^2; Q^n * S^2) = I_n(P^n; Q^n) + I_2(R^2; S^2) \tag{6}$$

Comparing (5) and (6), we obtain for $n = 2$ and 3,

$$\sum_{i=1}^{n} \phi(p_i r, p_i(1 - r); q_i s, q_i(1 - s)) = I_2(R^2; S^2) \tag{7}$$

In particular for $n = 3$, i.e., for $(P^3; Q^3) \in \Delta_3$,

$$\begin{aligned} &\phi(p_1 r, p_1(1 - r); q_1 s, q_1(1 - s)) \\ &\quad + \phi(p_2 r, p_2(1 - r); q_2 s, q_2(1 - s)) \\ &\quad + \phi(p_3 r, p_3(1 - r); q_3 s, q_3(1 - s)) = I_2(R^2; S^2) \end{aligned} \tag{8}$$

and also replacing the distribution (p_1, p_2, p_3) by $(p_1 + p_2, p_3)$ and (q_1, q_2, q_3) by $(q_1 + q_2, q_3)$, we have

$$\begin{aligned} &\phi((p_1 + p_2)r, (p_1 + p_2)(1 - r); (q_1 + q_2)s, (q_1 + q_2)(1 - s)) \\ &\quad + \phi(p_3 r, p_3(1 - r); q_3 s, q_3(1 - s)) = I_2(R^2; S^2) \end{aligned} \tag{9}$$

Subtracting (9) from (8), we have

$$\begin{aligned} &\phi((p_1 + p_2)r, (p_1 + p_2)(1 - r); (q_1 + q_2)s, (q_1 + q_2)(1 - s)) \\ &\quad = \phi(p_1 r, p_1(1 - r); q_1 s, q_1(1 - s)) \\ &\qquad + \phi(p_2 r, p_2(1 - r); p_2 s, p_2(1 - s)) \end{aligned}$$

which gives (4). Q.E.D.

Lemma 3. Let the function ϕ as given in Axiom 2 be (i) bounded and

(ii) $\psi_{r,s}(0; q) = 0 \quad \text{for} \quad q \in [0, 1]$ (10)

then

$$\phi(p_1, p_2; q_1, q_2) = (p_1 + p_2) I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) \tag{11}$$

where I_n satisfies Axioms 1–3.

Proof. Putting $p_1 = 0$ in (4) and using (10), we have

$$\psi_{r,s}(p_2; q_2) = \psi_{r,s}(p_2; q_1 + q_2) \tag{12}$$

for $p_2 \in [0, 1]$, $0 \leqslant q_2, q_1 + q_2 \leqslant 1$.

Thus we may say that $\psi_{r,s}(p; q)$ is independent of q. So we write

$$\psi_{r,s}(p_2; q) = b_{r,s}(p_2) \qquad \text{for} \quad p_2 \in [0, 1] \tag{13}$$

Equation (4) then becomes

$$b_{r,s}(p_1) + b_{r,s}(p_2) = b_{r,s}(p_1 + p_2) \tag{14}$$

for $p_1, p_2 \in [0, 1]$ with $p_1 + p_2 \leqslant 1$.

Since the function $\psi_{r,s}(p; q)$ is bounded, so is $b_{r,s}(p)$; therefore the solution of the Cauchy's functional equation (14) is given by

$$b_{r,s}(p) = p b_{r,s}(1) \tag{15}$$

But from (12), (13), and (15) (see Kannappan[8]) we find that

$$\psi_{r,s}(p; q) = p\psi_{r,s}(1; 1), \qquad p \in [0, 1], \quad q \in [0, 1]$$

In addition, the definition of $\psi_{r,s}(p; q)$ implies

$$\phi(pr, p(1 - r); qs, q(1 - s)) = p\phi(r, 1 - r; s, 1 - s) \tag{16}$$

$p, q \in [0, 1]$. Thus from (7) we get

$$I_2(R^2; S^2) = \phi(r, 1 - r; s, 1 - s) \tag{17}$$

Now setting $p_1 = pr$, $p_2 = p(1 - r)$, $q_1 = qs$, and $q_2 = q(1 - s)$ in (16) and using (17), we have Eq. (11), which proves the lemma. Q.E.D.

As a matter of consequence, Axiom 2 reduces precisely to

$$I_n(P^n; Q^n) - I_{n-1}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) = (p_1 + p_2) I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) \tag{18}$$

which is now the form of the branching property.

Lemma 4. If $v_k \geqslant 0$, $k = 1, 2, \ldots, m$, $\sum_{k=1}^{m} v_k = p_i > 0$, and $h_k > 0$, $k = 1, 2, \ldots, m$, $\sum_{k=1}^{m} h_k = q_i > 0$ for every $i = 1, 2, \ldots, n$, then

$$I_{m+n-1}(p_1, \ldots, p_{i-1}, v_1, \ldots, v_m, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, h_1, \ldots, h_m, q_{i+1}, \ldots, q_n)$$

$$= I_n(P^n; Q^n) + p_i I_m\left(\frac{v_1}{p_i}, \ldots, \frac{v_m}{p_i}; \frac{h_1}{q_i}, \ldots, \frac{h_m}{q_i}\right) \tag{19}$$

Proof. For $m = 2$ this reduces to (18). The lemma will be proved by induction.

Applying (19) for m in I_{m+n},

$$I_{m+n}(p_1, \ldots, p_{i-1}, v_1, \ldots, v_{m+1}, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, h_1, \ldots, h_{m+1}, q_{i+1}, \ldots, q_n)$$

$$= I_{n+1}(p_1, \ldots, p_{i-1}, v_1, \bar{p}, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, h_1, \bar{q}, q_{i+1}, \ldots, q_n)$$

$$+ \bar{p} I_m\left(\frac{v_2}{\bar{p}}, \ldots, \frac{v_{m+1}}{\bar{p}}; \frac{h_2}{\bar{q}}, \ldots, \frac{h_{m+1}}{\bar{q}}\right) \tag{20}$$

$$= I_n(P^n; Q^n) + p_i I_2\left(\frac{v_1}{p_i}, \frac{\bar{p}}{p_i}; \frac{h_1}{q_i}, \frac{\bar{q}}{q_i}\right)$$

$$+ I_m\left(\frac{v_2}{\bar{p}}, \ldots, \frac{v_{m+1}}{\bar{p}}; \frac{h_2}{\bar{q}}, \ldots, \frac{h_{m+1}}{\bar{q}}\right) \tag{21}$$

($\bar{p} = v_2 + \cdots + v_{m+1}$, $\bar{q} = h_2 + \cdots + h_{m+1}$) reducing I_{n+1} to I_n and I_2 by (18).

But for $n = 2$ and $m = m$, (19) is

$$I_{m+1}\left(\frac{v_1}{p_i}, \ldots, \frac{v_{m+1}}{p_i}; \frac{h_1}{p_i}, \ldots, \frac{h_{m+1}}{p_i}\right)$$

$$= I_2\left(\frac{v_1}{p_i}, \frac{\bar{p}}{p_i}; \frac{h_1}{q_i}, \frac{\bar{q}}{q_i}\right)$$

$$+ \left(\frac{\bar{p}}{p_i}\right) I_m\left(\frac{v_2}{\bar{p}}, \ldots, \frac{v_{m+1}}{\bar{p}}; \frac{h_2}{\bar{q}}, \ldots, \frac{h_{m+1}}{\bar{q}}\right) \tag{22}$$

Using (21) in (20), the result of the lemma follows for $m + 1$. Q.E.D.

Lemma 5. If $v_{ij} \geqslant 0$, $j = 1, 2, \ldots, m_i$, $\sum_{j=1}^{m_i} v_{ij} = p_i > 0$, and $h_{ij} > 0$, $j = 1, 2, \ldots, m_i$, $\sum_{j=1}^{m_i} h_{ij} \leqslant 1$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$, $\sum_{i=1}^{n} q_i \leqslant 1$, then

$$I_{nm_n}(V^{nm_n}; H^{nm_n})$$

$$= I_n(P^n; Q^n) + \sum_{i=1}^{n} p_i I_{m_i}\left(\frac{v_{i1}}{p_i}, \ldots, \frac{v_{im_i}}{p_i}; \frac{h_{i1}}{q_i}, \ldots, \frac{q_{im_i}}{q_i}\right) \tag{23}$$

This follows simply from the above lemmas (refer to Havrda and Charvat[5]).

Next if in Lemma 5 we replace m_i by m, $v_{ij} = 1/mn$, $h_{ij} = 1/rs$, $q_i = 1/s$, $i = 1, 2,..., n$, $j = 1, 2,..., m$, where m, n, r, and s are positive integers such that $1 \leqslant m \leqslant r$, $1 \leqslant n \leqslant s$, then we obtain

$$F(mn; rs) = F(m; r) + F(r; s) \tag{24}$$

where

$$F(m; r) = I(1/m,..., 1/m; 1/r,..., 1/r) \tag{25}$$

We now state without proof the standard result (see Aczel,[1] Chapter 5).

Lemma 6. The most general bounded solution of the Cauchy's functional equation in two variables given by (24) is

$$F(m; r) = A' \log m + B' \log r \tag{26}$$

where A' and B' are arbitrary constants.

Thus we can say that Lemmas 1–6 are consequences of axioms 1–3.
We now come to the central theorem of this paper.

Theorem 1. Axioms 1–3 together with the continuity of I_n in the region Δ_n determine the function I_n as

$$I_n(P^n; Q^n) = A \sum_{i=1}^{n} p_i \log p_i + B \sum_{i=1}^{n} p_i \log q_i \tag{27}$$

where A and B are arbitrary constants.

Proof. If m, r_i, and t_i are positive integers such that $\sum_{i=1}^{n} r_i = m$ and $\sum_{i=1}^{n} t_i = m$ and if we put $p_i = r_i/m$, $q_i = t_i/r$, $i = 1, 2,..., n$, then an application of Lemma 5 gives

$$I(1/m,..., 1/m; 1/r,..., 1/r)$$
$$= I_n(P^n; Q^n) + \sum_{i=1}^{n} p_i I(1/r_i,..., 1/r_i; 1/t_i,..., 1/t_i) \tag{28}$$

or

$$F(m; r) = I_n(P^n; Q^n) + \sum_{i=1}^{n} p_i F(r_i; t_i) \tag{29}$$

Thus (29) together with (26) and (28) gives Eq. (27), where $A = -A'$ and $B = -B'$ are arbitrary constants and then continuity of I_n proves the result for reals. Q.E.D.

3. APPLICATIONS TO INFORMATION THEORY

As remarked earlier, Kullback's information (or directed divergence) and Kerridge's inaccuracy are two information-theoretic measures associated with a pair of distributions and their characterizations are given below.

Theorem 2. (Kullback's information): The continuous mapping I_n: $\Delta_n \to R$ (reals) under Axioms 1–3 and with

$$I_2(P^2; P^2) = 0, \qquad p \in (0, 1) \tag{30}$$

and

$$I_2(1, 0; \tfrac{1}{2}, \tfrac{1}{2}) = 1 \tag{31}$$

is given by

$${}_1I_n(P^n; Q^n) = \sum_{i=1}^{n} p_i \log (p_i/q_i) \tag{32}$$

Proof. Equation (27) with (30) gives $A + B = 0$ and then (31) gives $A = 1$. Thus (27) becomes (32), which is Kullback's information. Q.E.D.

Theorem 3. (Kerridge's inaccuracy): The continuous mapping I_n: $\Delta_n = R$ (reals) under Axioms 1–3 and with

$$I_3(p_1, p_2, p_3; q_1, q_2, q_3) = I_2(p_1, p_2 + p_3; q_1, q_2) \tag{33}$$

and

$$I_2(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}) = 1 \tag{34}$$

is given by

$${}_2I_n(P^n; Q^n) = -\sum_{i=1}^{n} p_i \log q_i \tag{35}$$

Proof. Here Eq. (27) with (33) gives $A = 0$ and then (34) gives $B = -1$. Thus (27) reduces to (35), which is Kerridge's inaccuracy. Q.E.D.

Note. Condition (33) is Axiom 4 taken by Kerridge[9] and, as emphasized by Kerridge, is the most important additional axiom for characterizing inaccuracy.

4. GENERALIZED MEASURE OF TYPE (α, β)

Now let the mapping I_n denoted by $I_n^{(\alpha,\beta)}$ depend on two parameters α and β, and in place of Axiom 2 [form (18)], we have the branching property of type (α, β) given by:

Axiom 2′. (Generalized branching property):

$$\begin{aligned} &I_{n+1}^{(\alpha,\beta)}(p_1, \ldots, p_{i-1}, v_{i_1}, v_{i_2}, p_{i+1}, \ldots, p_n; \\ &\quad q_1, \ldots, q_{i-1}, h_{i_1}, h_{i_2}, q_{i+1}, \ldots, q_n) \\ &\quad = I_n^{(\alpha,\beta)}(P^n; Q^n) + p_i^{\alpha} q_i^{\beta} I_2\left(\frac{v_{i_1}}{p_i}, \frac{v_{i_2}}{p_i}; \frac{h_{i_1}}{q_i}, \frac{h_{i_2}}{q_i}\right) \end{aligned}$$

for every $v_{i_1} + v_{i_2} = p_i > 0$, $h_{i_1} + h_{i_2} = q_i > 0$, $i = 1, 2,..., n$, where α and β are arbitrary parameters such that $\alpha \neq 1$, $\beta \neq 0$.

It can be seen that Axiom 2′ reduces to (18) for $\alpha = 1$, $\beta = 0$.

This generalized branching property now gives measures whose characterization is given in the next theorem.

Theorem 4. Axioms 1, 2′, and 3 together with continuity of $I_n^{(\alpha,\beta)}$ in the region Δ_n determine the function $I^{(\alpha,\beta)}$ as

$$I_n^{(\alpha,\beta)}(P^n; Q^n) = C(\alpha, \beta)\left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta} - 1\right], \qquad \alpha \neq 1, \quad \beta \neq 0 \tag{36}$$

where $C(\alpha, \beta)$ ($\neq 0$) is a constant depending upon the parameters α and β.

Proof. With the help of Axiom 2′, Lemma 5 takes the form

$$I_{nm_n}^{(\alpha,\beta)}(V^{nm_n}; H^{nm_n})$$
$$= I_n^{(\alpha,\beta)}(P^n; Q^n) + \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta} I_{m_i}^{(\alpha,\beta)}\left(\frac{v_{i1}}{p_i},..., \frac{v_{im_i}}{p_i}; \frac{h_{i1}}{q_i},..., \frac{h_{im_i}}{q_i}\right) \tag{37}$$

Now setting the substitution given in (23) in (37), we obtain

$$F^{(\alpha,\beta)}(mn; rs) = F^{(\alpha,\beta)}(n; s) + (1/n)^{\alpha-1}(1/s)^{\beta} F^{(\alpha,\beta)}(m; r) \tag{38}$$

where $F^{(\alpha,\beta)}(m; r) = I^{(\alpha,\beta)}(1/m,..., 1/m; 1/r,..., 1/r)$.

Because of the symmetry of $I_n^{(\alpha,\beta)}$, (38) can be written as

$$F^{(\alpha,\beta)}(mn; rs) = F^{(\alpha,\beta)}(m; r) + (1/m)^{\alpha-1}(1/r)^{\beta} F^{(\alpha,\beta)}(n; s) \tag{39}$$

Equations (38) and (39) give

$$F^{(\alpha,\beta)}(m; r) = C(\alpha, \beta)[(1/m)^{\alpha-1}(1/r)^{\beta} - 1] \tag{40}$$

where $C(\alpha, \beta)$ ($\neq 0$) is a constant depending upon the parameters α and β.

Again setting (28) in (37), we obtain

$$I^{(\alpha,\beta)}(1/m,..., 1/m; 1/r,..., 1/r)$$
$$= I_n^{(\alpha,\beta)}(P^n; Q^n) + \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta} I^{(\alpha,\beta)}(1/r_i,..., 1/r_i; 1/t_i,..., 1/t_i)$$

i.e.,

$$I_n^{(\alpha,\beta)}(P^n; Q^n) = F^{(\alpha,\beta)}(m; r) - \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta} F^{(\alpha,\beta)}(r_i; t_i) \tag{41}$$

Now (41) together with (40) gives (36). Q.E.D.

Earlier Sharma and Ram Autar[12,13] have studied a quantity

$$I_n(P^n; Q^n) = (2^{\beta-\alpha} - 1)^{-1}\left[\sum_{i=1}^{n} p_i^{\beta} q_i^{\alpha-\beta} - 1\right], \qquad \alpha \neq 1, \quad \beta \neq 1$$

which arises from the study of generalized functional equation.

4.1. Particular Cases

Case I. Expression (36) together with (30) and (31) gives

$$_1I_n^{\alpha}(P^n; Q^n) = (2^{\alpha-1} - 1)^{-1}\left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1\right], \qquad \alpha \neq 1 \tag{42}$$

Quantity (42) was earlier studied by Rathie and Kannappan[11] and reduces to Kullback's information (32) in the limiting case $\alpha \to 1$.

Case II. Expression (36) together with (33) and (34) gives

$$_1I_n^{\beta}(P^n; Q^n) = (2^{-\beta} - 1)^{-1}\left[\sum_{i=1}^{n} p_i q_i^{\beta} - 1\right], \qquad \beta \neq 0 \tag{43}$$

Expression (43) reduces to (35) when $\beta \to 0$, which is Kerridge's inaccuracy.

Some interesting properties of expression (36) will be studied elsewhere.

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